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RESONANCE INTERACTIONS OF WAVES IN AN ICE CHANNEL†

A. V. MARCHENKO

Moscow

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The properties of low-amplitude surface waves propagating in an ice channel are investigated in the shallow-water approximation. The ice cover is modelled either by a rigid cap or by a thin elastic plate floating on a liquid surface. It is shown that an ice channel is a waveguide for surface waves. The dispersive properties of the natural oscillations of the liquid in the channel are investigated. The resonance velocities of the motion of the load on the channel surface, at which the amplitude of the forced oscillations of the liquid increases without limit in time, are determined. The decay instability of the natural oscillations of high harmonics with respect to waves of the first mode is demonstrated. The process is described by the standard equations for non-linear three-wave interaction. The investigations lead to the conclusion that critical modes of motion of a boat are realizable in an ice channel.

Linear inhomogeneities in an ice cover may serve as waveguides along which flexural-gravity edge waves propagate [1–6]. This problem has been investigated theoretically mainly for straight cracks and ice ridges. It has been observed [6] that a through crack in the ice cover is the limiting case of a channel whose width approaches zero. Hence an ice channel is also a waveguide for flexural-gravity waves. The waveguide properties of channels in the ice cover have never been studied before.

Studies of waveguide effects in hydrodynamics are usually concerned with changes in the spectral composition near the coast or with the trapping of surface-wave energy by underwater obstacles. In the case considered here, besides, one is also interested in various aspects of the influence of natural oscillations in the channel on a boat moving in it.

It is well known [7] that the wave resistance of a boat in a narrow channel with rigid walls and an open water surface is a non-monotone function of the boat's speed. As the boat moves, a flow is created in the vicinity of the boat's position and also at some distance ahead of it. If the boat's speed is less than the first critical velocity υ_1, the flow is directed against the boat's motion. If the boat's speed exceeds the second critical velocity υ_2, the directions of the flow and the boat's motion coincide. At speeds $\upsilon_1 < \upsilon < \upsilon_2$ the flow around the boat is of a complicated, non-steady nature.

The special resistance properties of the motion of boats in a channel are due to the presence of the walls. Waves diverging from the boat are reflected from the walls, their reflections hit the boat's hull, and this results in the superposition of wave systems that affect the wave resistance. In the case of an ice channel there are no walls. However, the continuous ice cover outside the channel reflects waves in the same way as walls do, since it possesses fairly strong elastic properties under bending deformations.

The main purpose of this purpose is to demonstrate the analogy between an ordinary channel with solid walls and an ice channel and to estimate the possibility of critical modes of motion of a boat in an ice channel.

1. THE NATURAL OSCILLATIONS OF A LIQUID IN AN ICE CHANNEL

Ice cover represented by a rigid cap. Consider a layer of shallow liquid of depth H under a solid cap ("ice cover") with a gap in the form of an infinite strip of width $2a$ with straight edges (the "channel"). The linearized equations of motion of the liquid are

$$\frac{\partial \eta}{\partial t} + H\Delta\varphi = 0, \quad \frac{\partial \varphi}{\partial t} + g\eta = 0, \quad |x| < a; \quad \Delta\varphi = 0, \quad |x| > a \tag{1.1}$$

where η is the perturbation of the liquid surface in the channel relative to the horizontal equilibrium position, φ is the velocity potential, and x, y and t are the horizontal coordinates and time, respectively.

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We will assume that the shallow-water approximation is applicable if $H/a < 1/2$. Equations (1.1) hold because the waves in question are of low amplitude.

In dimensionless variables, indicated by primes (which will be omitted later)

$$t = \frac{a}{\sqrt{gH}}t', \quad x = ax', \quad y = ay' \tag{1.2}$$

Eqs (1.1) may be written in the form

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi = 0, \quad |x| < 1; \quad \Delta\varphi = 0, \quad |x| > 1 \tag{1.3}$$

The solutions of Eqs (1.3) in the regions $|x| > 1$ and $|x| < 1$ must be related by conditions implied by the laws of conservation of mass and momentum

$$\lim\nolimits_{\pm}^{\pm} \frac{\partial^N \varphi}{\partial x^N} = \lim\nolimits_{\pm}^{\mp} \frac{\partial^N \varphi}{\partial x^N}, \quad N = 0, 1 \tag{1.4}$$

$$\left(\lim\nolimits_{\pm}^{+} = \lim_{x \to \pm 1 + 0}, \quad \lim\nolimits_{\pm}^{-} = \lim_{x \to \pm 1 - 0}\right)$$

We will investigate solutions corresponding to periodic waves travelling along the channel

$$\varphi = \varphi(x)e^{i(\gamma t + ky)}, \quad k > 0; \quad \varphi \to 0, \quad |x| \to \infty \tag{1.5}$$

From Eqs (1.3) we find the solution

$$\varphi(x) = C_1 e^{inx} + C_2 e^{-inx}, \quad |x| < 1 \tag{1.6}$$

$$\varphi(x) = C_- e^{kx}, \quad x < -1; \quad \varphi(x) = C_+ e^{-kx}, \quad x > 1; \quad n^2 = \gamma^2 - k^2$$

Substituting expressions (1.6) with the four constants $C_{1,2}$, $C_\pm$ into (1.4), we obtain an algebraic system of four linear homogeneous equations for $C_{1,2}$, $C_\pm$

$$C_\pm e^{-k} = C_1 e^{\pm in} + C_2 e^{\mp in}, \quad \mp C_\pm e^{-k} = in(C_1 e^{\pm in} - C_2 e^{\mp in}) \tag{1.7}$$

This system has solutions corresponding to waves which are symmetric and anti-symmetric about the plane $x = 0$ and satisfy the conditions $C_+ = C_-$, $C_1 = C_2$ and $C_+ = -C_-$, $C_1 = -C_2$, respectively. The condition for Eqs (1.7) to have non-trivial solutions is that the determinant should vanish, which may be represented as a product $\Delta_s(\gamma, k)\Delta_{as}(\gamma, k)$, where

$$\Delta_s = n \sin n - k \cos n, \quad \Delta_{as} = n \cos n + k \sin n \tag{1.8}$$

correspond to the determinants of the systems obtained from (1.7) for symmetric and anti-symmetric waves.

The solid curves in Fig. 1 are dispersion curves corresponding to non-trivial solutions. In view of the symmetry, only the first quadrant of the k, γ plane is shown. Even (odd) j values correspond to symmetric (anti-symmetric) waves. The dispersion curves have the following asymptotic form as $|k| \to 0$

$$\gamma_1 \approx \sqrt{k}, \quad \gamma_j \approx \frac{(j-1)\pi}{2} + \frac{k}{[(j-1)\pi]^2}, \quad j > 1 \tag{1.9}$$

As $k \to \infty$ all the curves asymptotically approach the straight line $\gamma = k$. In dimensional variables, in the limit as $a \to 0$, the curve γ_1 coincides with the frequency axis. The other curves γ_i go off to infinity in the γ direction.

The dashed curves in Fig. 1 are dispersion curves for natural oscillations in a channel with rigid side walls described by the equations $\sin n = 0$ and $\cos n = 0$, for symmetric and anti-symmetric waves. These relationships are readily obtained by substituting the first relation of (1.6) into the boundary

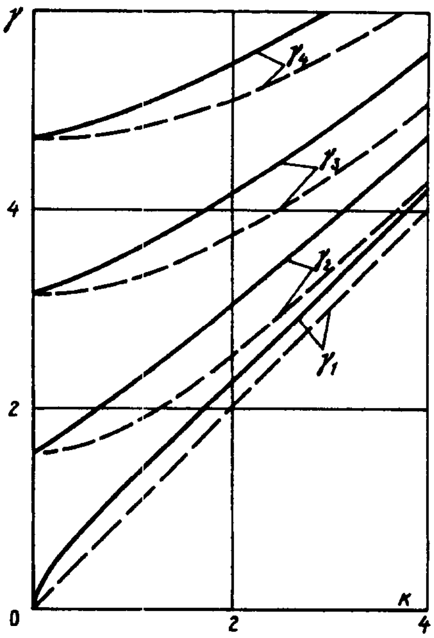

Fig. 1.

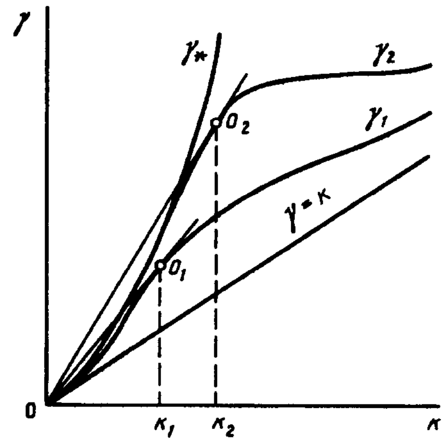

Fig. 2.

conditions $\partial\varphi/\partial x = 0$ for $|x| = 1$. In this case the first mode corresponding to a symmetric wave is a plane wave propagating along the y axis in a liquid with a free surface.

It can be seen from (1.6) that the solution for natural oscillations in a channel is a superposition of plane waves propagating in a liquid with a free surface in different directions of the x axis. Clearly, the dispersion curves of the first modes of vibration in an ice channel and in a channel with side walls differ significantly only in the low-frequency region. The group velocity of the natural oscillations in the first case is greater than in the second case.

Ice cover represented by an elastic plate. We consider a layer of shallow liquid under an elastic plate with a gap having the shape of an infinite strip with straight edges. In dimensionless variables (1.2) the equations of motion of the liquid may be written in the form

$$\left(\frac{\partial^2}{\partial t^2}-\Delta\right)\varphi=0, \quad |x|<1; \quad \left(\frac{\partial^2}{\partial t^2}-\Delta-D\Delta^3\right)\varphi=0, \quad |x|>1 \tag{1.10}$$

$$D=\frac{Eh^3}{12\rho g(1-\nu^2)a^4}$$

where E, ν and h are Young's modulus, Poisson's ratio and the thickness, respectively, of the elastic plate, and ρ is the density of the liquid.

The solution of (1.10) must satisfy the laws of conservation (1.4) and must be such that the transverse forces and bending moments acting at the edge of the elastic plate vanish [1]

$$\lim{}_{\pm}^{\pm}\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2}\right)\eta=0 \qquad \lim{}_{\pm}^{\pm}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\nu'\frac{\partial^2}{\partial y^2}\right)\eta=0, \quad \nu'=2-\nu \tag{1.11}$$

We will investigate solutions of problem (1.10), (1.4), (1.11) that satisfy conditions (1.5). Substituting (1.5) into (1.10), we find that when $|x| < 1$ the solution is given by formulae (1.6), but when $|x| > 1$ we have

$$\varphi(x)=\sum_{j=1}^{3} C_j^{\pm}e^{\pm i\lambda_j x} \tag{1.12}$$

The plus and minus signs in (1.12) correspond to the regions $x > 1$ and $x < -1$. The numbers $\lambda_j = \lambda_j(\gamma, k)$ are the roots of the dispersion equation $\gamma^2 = (k^2 + \lambda^2)[1 + D(k^2 + \lambda)^2]^2$ that satisfy the condition $\text{Im}\,\lambda_j > 0$. This is satisfied when $\gamma^2 < \gamma_*^2 = k^2(1 + Dk^4)$.

To determine the eight unknown constants $C_j^{\pm}$, $C_{1,2}$, we derive from (1.4) and (1.11) the following system of eight homogeneous linear algebraic equations

$$\sum_{j=1}^{3} C_j^{\pm}(\lambda_j^2 + \nu k^2)(\lambda_j^2 + k^2)e^{i\lambda_j} = 0$$
$$\sum_{j=1}^{3} C_j^{\pm}\lambda_j(\lambda_j^2 + \nu' k^2)(\lambda_j^2 + k^2)e^{i\lambda_j} = 0 \qquad (1.13)$$
$$\sum_{j=1}^{3} C_j^{\pm}e^{i\lambda_j} = C_1 e^{\pm in} + C_2 e^{\mp in}$$

$$\sum_{j=1}^{3} C_j^{\pm}\lambda_j e^{i\lambda_j} = \pm n(C_1 e^{\pm in} - C_2 e^{\mp in}) \qquad (1.14)$$

The solutions of Eqs (1.13) and (1.14) for symmetric and anti-symmetric waves have the form $C_+ = C_-$, $C_j^+ = C_j^-$ and $C_+ = -C_-$, $C_j^+ = -C_j^-$, respectively. We will not present explicit expressions for the determinants $\Delta_s(\gamma, k)$ and $\Delta_{as}(\gamma, k)$ of the matrices of order four representing symmetric and anti-symmetric waves.

Numerical investigations show that the dispersion curves described by the equations $\Delta_s \doteq 0$ and $\Delta_{as} = 0$ have the form shown in Fig. 2, where even and odd j correspond to symmetric and anti-symmetric waves, respectively. The curve γ_1 originates from the origin. The initial point of the other curves γ_j lies ono the curve γ_*. As $k \to \infty$ all the curves tend to the straight line $\gamma = k$.

It is easy to see that in dimensional variables, as $a \to 0$, the initial points of all the dispersion curves for $j > 1$ go to infinity. In this limiting case the curve γ_1 corresponds to a symmetric edge wave propagating along a crack in an elastic plate floating on a liquid surface [1]. The dispersion curve for this edge wave is represented in Fig. 3 by the curve γ_s.

Note that one other edge wave has been observed [1], corresponding to an anti-symmetric mode. This is due to the substitution $\nu = \nu'$ in the contact-boundary conditions (1.11) [1].

Consider a liquid of finite depth H. In that case the dispersion relations will depend on the dimensionless parameter a/H, which was previously equated to zero. In what follows we will denote the dimensionless parameter H/a by H. It has been shown [4] that the dispersion curve for a symmetric edge wave $\gamma_1(k)$ travelling along a crack in an elastic plate has the form represented in Fig. 3 by the dashed curve γ_s^H. In a liquid of finite depth, the curve y_*^H represents plane waves travelling along the crack. There is no anti-symmetric edge mode.

If the dimensionless depth of the liquid is sufficiently small, the points of inflection of the dispersion curves γ_s^H, γ_*^H lie in a region where the shallow-water approximation holds. Therefore, the dispersion

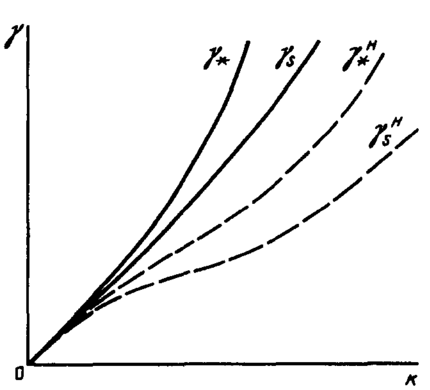

Fig. 3.

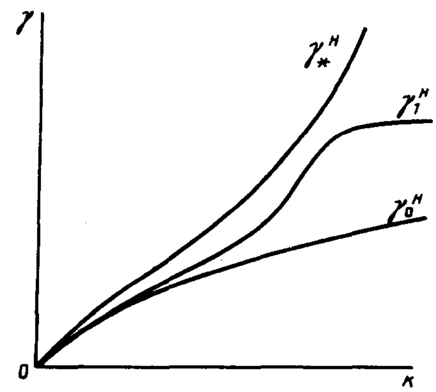

Fig. 4.

curve γ_s^H, describing the first mode of a symmetric wave in the channel in a liquid of finite depth, may have the form shown in Fig. 4. It can be seen that γ_s^H has two points of inflection. As $k \to \infty$ the curve γ_1 asymptotically approaches the dispersion curve γ_0^H for waves in a liquid of finite depth, where $(\gamma_0^H)^2 = k\,\mathrm{th}\,(kH)$. The initial point of the other curve for higher oscillation modes in the channel will lie, as before, on the y_*^H curve.

2. RESONANCE EXCITATION OF NATURAL OSCILLATIONS

We will consider the problem of exciting waves in a channel which travel at a constant velocity V and have an oscillating pressure field $P(y - Vt, x)e^{i\omega t}$. The equations of motion of the liquid in a reference frame moving at velocity V are, in dimensionless variables

$$\left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial \xi}\right)\eta + \Delta\varphi = 0, \quad -\infty < x < +\infty$$

$$\left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial \xi}\right)\varphi + \eta + P(\xi, x)e^{i\omega t} = 0, \quad |x| < 1 \tag{2.1}$$

$$\left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial \xi}\right)\varphi + \eta + D\Delta^2\eta = 0, \quad |x| > 1$$

$$\xi = y - Vt, \quad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial x^2}$$

It is assumed that the motion begins at a time $t = 0$ from a state of rest

$$\eta = 0, \quad \varphi = 0, \quad t = 0 \tag{2.2}$$

At $t > 0$ waves will be excited in the channel. Since the wave velocity is finite, the region occupied by waves at any finite instant of time will be finite. We may therefore replace the functions φ and η by their Fourier–Laplace transforms with respect to ξ and t

$$(\zeta, \psi)(l, k, x) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \int_{-\infty}^{\infty} d\xi e^{-lt + ik\xi}(\eta, \varphi)(t, \xi, x)$$

The Fourier transforms ψ and ζ should not have singularities near the real axis k, and the contour of integration may be deformed in its neighbourhood. In order to analyse the Fourier solution, we split the integrals into a sum, where each term corresponds to waves of a certain type. The integrand may then have singularities on the real k axis. The deformation of the contour in the general solution of [8] was necessary to ascertain the asymptotic behaviour at long times t of forced waves propagating at the velocity of motion of an external load.

It follows from (2.1) and (2.2) that

$$(l - ikV)\zeta + \left(\frac{\partial^2}{\partial x^2} - k^2\right)\psi = 0$$

$$(l - ikV)\psi + \zeta + D\left(\frac{\partial^2}{\partial x^2} - k^2\right)^2\psi = 0, \quad |x| > 1 \tag{2.3}$$

$$(l - ikV)\psi + \zeta + \frac{P_f(k, x)}{l - i\omega} = 0, \quad |x| < 1$$

In what follows we will confine our attention to the case in which P_f is independent of x.

The solution of Eqs (2.3) for $|x| > 1$ is given by formulae (1.12), where $\lambda_j = \lambda_j(l - ikV, k)$ are the roots of the equation

$$(l - ikV)^2 + (k^2 + \lambda^2)[1 + D(k^2 + \lambda^2)^2] = 0$$

that satisfy the condition $\mathrm{Im}\lambda_j > 0$.

The solution of Eqs (2.3) for $|x| < 1$ has the form

$$\psi(l,\ k,\ x) = C_1 e^{-mx} + C_2 e^{mx} - \frac{P_f(l-ikV)}{(l-i\omega)m^2},\quad m^2 = (l-ikV)^2 + k^2 \tag{2.4}$$

Substituting (1.12) and (2.4) into Eqs (1.4) and (1.11), we obtain an inhomogeneous system of eight linear algebraic equations for the constants $C_{1,2}$, $C_j^{\pm}$. Four of them are identical with (1.13). The other four equations are

$$\sum_{j=1}^{3} C_j^{\pm} e^{i\lambda_j} = \psi|_{x=\pm 1},\quad \sum_{j=1}^{3} C_j^{\pm} e^{i\lambda_j} = \mp i \frac{\partial\psi}{\partial x}\Big|_{x=\pm 1} \tag{2.5}$$

The solution of Eqs (1.13) and (2.5) is

$$(C_{1,2},\ C_j^{\pm}) = \frac{P_f(l-ikV)(E_{1,2},\ E_j^{\pm})}{\Delta(l-i\omega)m^2}$$

where $E_{1,2}$, $E_j^{\pm}$, Δ are functions of $l - ikV$ and k, and $\Delta(l - ikV,\ k)$ is the determinant of the system of equations (1.13) and (1.14) after making the substitution $l - ikV \to i\gamma$.

Performing the inverse Fourier and Laplace transformation, we obtain

$$\varphi = \frac{P_f}{i(2\pi)^{3/2}} \int_{-i\infty}^{i\infty} dl \int_{-\infty}^{\infty} dk \frac{(l-ikV)e^{lt-ik\xi}}{\Delta(l-i\omega)m^2}(E_1 e^{-mx} + E_2 e^{mx} + \Delta)$$

The solution for $|x| > 1$ is determined in a similar way.

The integrals in the last expression are evaluated by residues in the planes of the complex variables l and k. We will consider the residues corresponding to the zeros of the functions $\Delta(l - ikV,\ k)$. The zeros of Δ on the complex l axis correspond to the natural oscillations in the channel. Hence, in the neighbourhood of each zero we can write

$$\Delta = (l - ikV + i\gamma_j(k))\Delta'$$

Denote the part of the solution corresponding to the residue $l = l_j \equiv i(kV - \gamma_j(k))$ by φ_j. We find from the formula for the solution that

$$\varphi_j = \frac{P_f}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dk (E_1 e^{-mx} + E_2 e^{mx})|_{l=l_j} \frac{\gamma_j \exp(i[(kV-\gamma_j)t - k])}{\Delta'(\gamma_j,\ k)(kV - \gamma_j - \omega)(\gamma_j^2 - k^2)} \tag{2.6}$$

Let us investigate the asymptotic behaviour of the integral (2.6) at high t values. In the neighbourhood of $k = k_0$ we rewrite the expression $kV - \gamma_j - \omega$ as

$$kV - \gamma_j - \omega = (k-k_0)(V - \gamma_j'(k_0)) + k_0 V - \gamma(k_0) - \omega + O((k-k_0)^2)$$

$$\gamma_j'(k_0) = \frac{\partial\gamma_j}{\partial k}\Big|_{k=k_0}$$

Suppose that the pressure field is moving at the group velocity of the natural mode of the waveguide with wave number k_0 in the direction of y, that is, the following equation is satisfied

$$V = \gamma_j'(k_0) \tag{2.7}$$

and the oscillation frequency of the external pressure is such that the frequency of natural oscillations of the external pressure field equals the frequency of the wave in a reference frame attached to the moving load

$$\omega = \gamma(k_0) - k_0 V \tag{2.8}$$

It follows from (2.7) and (2.8) that the integrand in (2.6) has a singularity of the type $(k-k_0)^2$ in the denominator, while its numerator is proportional to exp $[i(k-k_0)^2t]$. The principal contribution to the asymptotic behaviour of the solution for large t comes from an integral of type (2.6) around a contour C in the neighbourhood of $k = k_0$. The contour C in the complex k plane consists of two half-lines that approach the point $k = k_0$. The contour C in the complex k plane consists of two half-lines that approach the point $k = k_0$ at an angle $\pi/4$ and describe a semicircle around $k = k_0$ [8]. It has been shown that

$$\int_C f(z)\frac{e^{iz^2t}}{z^2}dz = 2f(0)e^{-i\pi/4}(\pi t)^{1/2} + i\pi f'(0) + O(t^{-1/2})$$

Thus, as $t \to \infty$ the integral φ_j is proportional to exp $(i\omega t)\sqrt{t}$. If the velocity of the load equals the external phase velocity of natural oscillations in the channel, then the integral (2.6) will increase in proportion to $\sqrt{t}$, and $\omega = 0$. In what follows we will call these "resonance velocities". The wave numbers of waves with minimum phase velocity are denoted by k_j in Fig. 2. At these points the resonance velocity V equals the slope of the tangent through the origin to the dispersion curve γ_j. The numbers k_j are functions of a single dimensionless parameter D.

It can be seen that in a liquid of finite depth one further resonance velocity, equal to $\sqrt{(gH)}$ may appear, since a natural wave moving at this velocity has an extremal phase velocity. This phase velocity is a minimum. Thus, unlike the case of a continuous ice cover [8], when there are only two resonance velocities, when a load is moving on the surface of a channel in the ice cover there are several resonance velocities.

Note that conditions (2.7) and (2.8) may be satisfied when one is investigating the two-dimensional problem of a vibrating load moving on the surface of a liquid layer with a free surface. In that case the amplitude of the forced oscillations will also increase with time in proportion to $\sqrt{t}$. In the three-dimensional case resonance phenomena may disappear owing to two-dimensional dispersive effects.

3. RESONANCE INTERACTIONS OF WAVES IN A CHANNEL

It was shown above that the natural oscillations of a liquid in a channel have a dispersion relation with several branches. It is of interest to study the influence of non-linear effects on the possible interactions between waves corresponding to different branches. As will be seen below, for any wave in a higher mode, waves in the first mode exist to which energy is transferred as time passes. For simplicity, we will consider the case of a channel in a rigid cap floating on a liquid surface (Section 1).

The conditions for three-wave resonance interaction are

$$\gamma_a(k_3) = \gamma_b(k_2) + \gamma_c(k_3), \quad k_3 = k_1 + k_2 \tag{3.1}$$

where $\gamma_{a,b,c}(k)$ are the branches of the dispersion relation (1.8), which can be conveniently rewritten as follows:

$$\Delta(\gamma, k) \equiv e^{-in}(|k|-in) + (-1)^{\alpha}e^{in}(|k|+in) = 0 \tag{3.2}$$

Even (odd) α correspond to symmetric (anti-symmetric) waves.

Resonance triads are easily constructed graphically, using the well-known procedure of [9]. We select an arbitrary point A with coordinates (γ_A, k_A) on the dispersion curve (DC) of the first mode corresponding to a symmetric wave. We will study the various resonance interactions that involve the wave (γ_A, k_A). To do so, we displace the origin to A and construct new DCs through A. These are shown in Fig. 5 by dashed curves. Let $C_1, C_2, \ldots$ be the points at which the dashed DC intersects the first branch γ_1 with the solid DCs. A straight line through the origin parallel to AC_j has a point of intersection, say B_j, with the solid DC γ_1. It is obvious that $OB_j = AC_j$. Hence it follows that

$$\gamma_{C_j} = \gamma_A + \gamma_{B_j}, \quad k_{C_j} = k_A + k_{B_j} \tag{3.3}$$

The points B_j and C_j have coordinates (γ_{B_j}, k_{B_j}) and (γ_{C_j}, k_{C_j}), respectively. Obviously, formulae (3.3) are analogous to conditions (3.1). Thus, a wave in each high-frequency mode has a property of decay instability with respect to waves in the first low-frequency mode travelling in opposite directions. We can also consider resonance interactions between higher-mode waves.

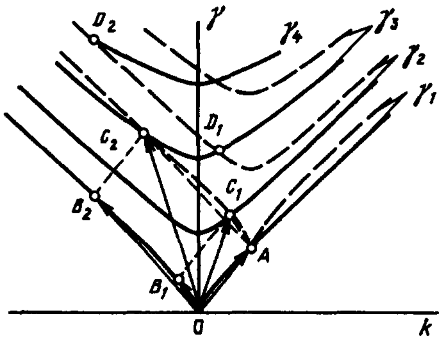

Fig. 5.

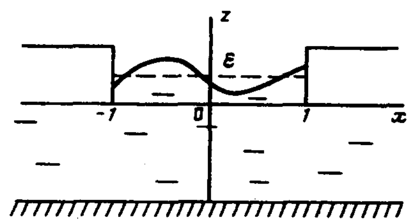

Fig. 6.

We now consider the derivation of the equations of non-linear resonance interaction of wave packets. In dimensionless variables, the non-linear shallow-water equations are

$$\frac{\partial\eta}{\partial t}+\varepsilon\nabla((\eta+1)\nabla\varphi)+\Delta\varphi=0,\quad \frac{\partial\varphi}{\partial t}+\frac{\varepsilon}{2}(\nabla\varphi)^2+\eta=0 \qquad |x|<1,\quad \varepsilon\ll 1 \tag{3.4}$$

When $|x|>1$ the potential φ satisfies Laplace's equation. The small parameter ε equals the ratio of the depth of submersion of the plate to a characteristic horizontal scale a, equal to half the width of the channel (Fig. 6).

By the law of conservation of mass

$$\lim\nolimits_{\pm}^{\pm}\frac{\partial\varphi}{\partial x}=\lim\nolimits_{\pm}^{\mp}\left[\frac{\partial\varphi}{\partial x}(1+\varepsilon+\varepsilon\eta)\right] \tag{3.5}$$

It follows from (3.5) and the pressure continuity condition on the straight line $z=0$ (Fig. 6) that the velocity potential φ is continuous at $|x|=1$

$$\lim\nolimits_{\pm}^{\pm}\varphi=\lim\nolimits_{\pm}^{\mp}\varphi+O(\varepsilon^2) \tag{3.6}$$

(allowance has been made for the fact that the velocity jumps for a liquid particle along the direction y at $|x|=1$ are of the order of ε^2).

Equations (3.4) and (3.6), considered to within 0/ε, yield the following problem of determining the potential φ

$$\left(\frac{\partial^2}{\partial t^2}-(1+\varepsilon)\Delta\right)\varphi+\varepsilon\frac{\partial}{\partial t}\left[(\nabla\varphi)^2+\frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right]=0,\quad |x|<1$$
$$\Delta\varphi=0,\quad |x|>1 \tag{3.7}$$

$$\lim\nolimits_{\pm}^{\pm}\frac{\partial\varphi}{\partial x}=\lim\nolimits_{\pm}^{\mp}\left[\frac{\partial}{\partial x}\left(1+\varepsilon-\varepsilon\frac{\partial\varphi}{\partial t}\right)\right],\quad \lim\nolimits_{\pm}^{\pm}\varphi=\lim\nolimits_{\pm}^{\mp}\varphi \tag{3.8}$$

We will seek the solution of (3.7) in the form

$$\varphi=\sum_{j=1}^{3}\varphi_j(T,Y,x)e^{i\theta_j}+\text{c.c.}+O(\varepsilon) \tag{3.9}$$
$$\theta_1=\gamma_b t+k_1 y,\ \theta_2=\gamma_c t+k_2 y,\ \theta_3=\gamma_a t+k_3 y,\ T=\varepsilon t,\ Y=\varepsilon y$$

The quantities $\gamma_{a,b,c}$ and $k_{1,2,3}$ satisfy conditions (3.1).

We deduce from (3.7) and (3.8) that, to within terms of order zero in ε

$$\varphi_j = \varphi_{j0}(T, Y)(e^{in_jx} + (-1)^{\alpha_j} e^{-in_jx}) + \varepsilon\varphi_{j1} + O(\varepsilon^2), \quad |x| < 1$$
$$\varphi_j = \Psi_j^{\pm} e^{\mp|k_j|x} + \varepsilon\varphi_{j1} + O(\varepsilon^2), \quad |x| > 1 \tag{3.10}$$
$$n_1 = \sqrt{\gamma_b^2 - k_1^2}, \quad n_2 = \sqrt{\gamma_c^2 - k_2^2}, \quad n_3 = \sqrt{\gamma_a^2 - k_3^2}$$

where $\alpha_j = 0$ for symmetric waves and $\alpha_j = 1$ for anti-symmetric waves. The plus and minus signs in the superscripts in the second formula of (3.10) correspond to the regions $x > 1$ and $x < -1$.

In the first approximation with respect to ε, we use (3.7) and the zeroth approximation (3.8) to find φ_{j1} and, after substituting expressions (3.10) into boundary conditions (3.8), we obtain

$$iL_j\varphi_{j0} = W_jN_j, \quad j = 1, 2, 3; \quad L_j = \left.\frac{\partial\Delta}{\partial\gamma}\right|_{k=k_j}\frac{\partial}{\partial T} + \left.\frac{\partial\Delta}{\partial k}\right|_{k=k_j}\frac{\partial}{\partial Y}$$

$$N_1 = \varphi_{20}^*\varphi_{30}, \quad N_2 = \varphi_{10}^*\varphi_{30}, \quad N_1 = \varphi_{10}\varphi_{20}$$

The coefficients W_j are functions of $\gamma_{a,b,c}$ and k_j, but the formulae are cumbersome and will not be given here. The numerical expressions W_j have been analysed numerically for specific resonance triads.

The relation

$$\left.-\frac{\partial\Delta}{\partial k}\left(\frac{\partial\Delta}{\partial\gamma}\right)^{-1}\right|_{k=k_j} = c_j$$

is equal to the group velocity of the wave packet φ_{j0} along the y axis. Thus, we have shown that the resonance interaction of natural oscillations in the channel is described by the standard equations of three-wave interaction, the properties of whose solutions have been thoroughly investigated [10]. It has been shown that complete decay of the pumping wave φ_{30} occurs if its group velocity is such that $c_3 \in (c_1, c_2)$. In the problem under consideration, this is the case if the waves φ_{10} and φ_{20} travel in opposite directions.

There is no allowance for dispersion in Eqs (3.7), and it can therefore be shown that the waves must break. However, it follows from conditions (3.8) that the processes by which higher harmonics are generated are not resonance processes, since oscillations with multiple frequencies and wave numbers along the y axis are not natural waves in the channel. Hence the waves do not break in this case.

4. CONCLUSIONS

The investigations described in this paper show that there is a profound analogy between natural oscillations of a liquid in an ordinary channel and in an ice channel. As a boat moves through an ice channel, some of the energy is radiated to infinity as flexural-gravity waves, and part of the boat's energy is used to produce natural oscillations in the channel. In the limiting case, when the rigidity of the ice cover is infinitely large (the approximation of a solid cap), there are no flexural-gravity waves and all the boat's energy is expended in producing natural oscillations and overcoming friction.

In an ordinary channel, the onset of the critical regime is due to the mutual influence of natural oscillations. One might expect that in an ice channel, as the velocity of motion increases, there should again be a tendency for the water level near the boat to fall and for a solitary wave to form ahead of the boat. A solitary wave would obviously represent a non-linear formation of natural oscillations of the first mode. By analogy with an ordinary channel, we estimate the critical velocity υ_1 [7] as being in the range $0.55\sqrt{(gH)} < \upsilon_1 < \sqrt{(gH)}$. Setting $H = 10$ m and $H = 50$ m, we obtain estimates for the first critical velocity: 5.4 m/s $< \upsilon_1 <$ 10 m/s and 12 m/s $< \upsilon_1 <$ 22 m/s, respectively. The typical speeds of boats in ice channels do not exceed 8 m/s. Hence it is clear that critical modes of motion can only occur in very shallow water.

Note that the wave resistance of a boat moving in a channel may increase because of the influence of natural oscillations on the hull. Typical periods of such oscillations lie in the range from 1 to 10 s, that is, $\omega \in (0.6, 6)\ \mathrm{s}^{-1}$.

Using formulae (2.7) and (2.8), we can estimate the speeds of a boat at which the amplitude of forced waves in the first mode of natural oscillations in the channel increases. It has been shown that the dispersion curve of the first mode differs only slightly from the dispersion curve of plane waves propagating in a liquid with a free surface. We may therefore set $\gamma_j(k) \approx \gamma_0(k) = \sqrt{(gk\,\mathrm{th}\,(kH))}$ in (2.7) and (2.8) to obtain estimates.

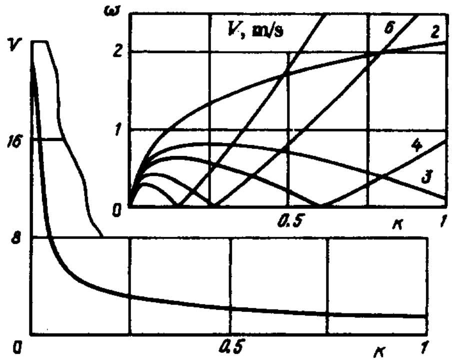

Fig. 7.

In the part of Fig. 7 plotted in the V, k plane, the vertical axis represents the velocity of motion V in m/s, and the horizontal axis is the wave number k in m^{-1}. The curve shown in the V, k plane corresponds to the group velocity of a plane wave $\gamma_0(k)$ at $H = 50$ m. It can be seen that the wave number of a wave with group velocity less than 8 m/s is greater than 0.04 m^{-1}.

In the part of Fig. 7 plotted in the ω, k plane, the vertical axis represents the frequency ω in s^{-1}, and the horizontal axis is the wave number k in m^{-1}. The curves shown in Fig. 7 in the ω, k plane are described by the equations $\omega = |kV - \sqrt{(gk \text{th}\,(kH))}|$ for $H = 50$ m and various values of V. It can be seen that when $k > 0.04$ m^{-1} there are always velocities less than 8 m/s at which $\omega \in (0.6, 6)$ s^{-1}. We have thus shown that there are always speed of motion of an external load, within the range of possible speeds of the boat, and corresponding frequencies in the range of characteristic frequencies of vibrations of the hull, at which the first vibration mode will be produced in the channel.

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